The problem of nonlinear diagonalisation

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Abstract

Although the tools of linear algebra are very useful in many fields, the linear structure is not always present. However, some of its results can be extended and generalised to broader contexts. In this work we present a concept analogous to the finite-dimensional diagonalisable linear maps for the non-linear case. After presenting our conceptual proposal, we analyse how some of the best-known results in this area carry over.

Introduction

Recall that diagonalising a linear map $T\colon \mathbb{R}^n \to \mathbb{R}^n$ consists in finding a basis in which the matrix expression of T is diagonal. In other words, the aim is to obatin a proper representation in which the map T is as simple as possible. The advantages are obvious, which is why diagonalisation is a classic problem in mathematics that has been studied extensively. This is why we are interested in the study of this concept in a more general setting in the nonlinear case. In this work we will show our approach to this problem and present some potential applications. The main references for this are [1] and [2].

Notation and definitions

In the finite dimensional Euclidean space $\mathbb{R}^n$ we denote the elements of the canonical basis as $e_1, \dots, e_n$. To refer to the vector

$$x = x_1 e_1 + \ldots + x_n e_n \in \mathbb{R}^n$$

we use the standard notation $(x_1, \dots, x_n)$. For a map $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ we write its image in terms of its coordinates in the canonical basis as $\Phi(x) = (\Phi_1(x), \dots, \Phi_n(x))$.
We say that the map Φ is *diagonal* if holds

$$\Phi(x_1, \dots, x_n) = (\Phi_1(x_1), \dots, \Phi_n(x_n)).$$

That is, the i-th coordinate of the image of x only depends of the i-th coordinate of x in the basis. If the map Φ is $\mathcal{C}^1$, then being diagonalisable is equivalent to the Jacobian matrix $D\Phi$ being diagonal. And Φ is *diagonalisable* if there exists a linear isomorphism $L\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\Psi := L^{-1} \circ \Phi \circ L$ is diagonal.

Results

Although the general definition of a diagonalisable function is intuitive, it is not practical as it cannot be easily checked. We therefore present some results that usefully characterise it. We also present a study of how some of the well-known definitions and results in the classical diagonalisation problem can also be extended and studied.

Characterizations

The following results characterizes the maps that can be diagonalised in the general case.

Theorem 1. *Given a function $\Phi\colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ being $\mathcal{C}^1$, it is diagonalisable if and only if $\{D\Phi(x)\}_{x\in\Omega}$ is a commuting family of diagonalizable Jacobian matrices.*

However, we present another characterization result for $\mathbb{R}^2$ which can be more practical since it will provide the derided basis.

Theorem 2. *Let $\Phi : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a $\mathcal{C}^1$ function. It is diagonalisable if and only if*

$$\Delta = (\varphi_{11} - \varphi_{22})^2 + 4\varphi_{21}\varphi_{12} \geq 0$$

and

$$\begin{aligned} A &= \frac{(\varphi_{11} - \varphi_{22})^2 + 2\varphi_{21}^2 + 2\varphi_{21}\varphi_{12}}{(\varphi_{11} - \varphi_{22})^2 + (\varphi_{12} + \varphi_{21})^2} \\ B &= \frac{(\varphi_{11} - \varphi_{22})^4 + 4(\varphi_{11} - \varphi_{22})^2\varphi_{21}\varphi_{12}}{\left((\varphi_{11} - \varphi_{22})^2 + (\varphi_{12} + \varphi_{21})^2\right)^2} \end{aligned} \tag{1}$$

are constant numbers (independent from x and y), except when $\varphi_{1,1} = \varphi_{2,2}$ and $\varphi_{1,2} = \varphi_{2,1} = 0$. Naming $t_1 = \frac{A+\sqrt{B}}{2}$ and $t_2 = \frac{A-\sqrt{B}}{2}$, the matrix of the isomorphism that diagonalises Φ is

$$P = \begin{pmatrix} \sqrt{1-t_1} & \sqrt{1-t_2} \\ \sqrt{t_1} & \sqrt{t_2} \end{pmatrix}.$$

Eigenvectors/Eigenfunctions

We can also define the related concepts to linear diagonalisation of eigenvector and eigenvalue. For $\Phi\colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$, we say that $u \in \mathbb{R}^n$ is an eigenvector of Φ if there exists $\lambda\colon \mathbb{R} \to \mathbb{R}$ such that

$$\Phi(hu) = \lambda(h)u,$$

for all $h \in \mathbb{R}$ such that $hu \in \Omega$. That is, Φ is invariant under the direction of u. We call λ an eigenfunction of Φ.
The relationship between the diagonalisation problem and the eigenvectors/eigenfunctions is weaker than in the linear case. In the linear case, if there exists a set of n linearly independent eigenvectors of a function T, then it is known that T is diagonalizable in the basis defined by these vectors. However, this is not the case if T is nonlinear. Indeed, the function $\Phi(x,y) = (y^2, xy)$ has linearly independent eigenvectors $(1,1)$ and $(1,-1)$, but it is not diagonalisable since $A = \frac{x^2+6y^2}{x^2+9y^2}$ is not constant. Nevertheless, it is possible to get a converse of this result assuming that the function is origin preserving.

Proposition 3. *Let $\Phi\colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a function such that $\Phi(0) = 0$. If Φ is diagonalizable, then there exist n linearly independent eigenvectors u_i and n eigenfunctions $\lambda_i\colon \mathbb{R} \to \mathbb{R}$, not necessarily different.*

Orthogonal diagonalisation

A classical result for symmetric matrices is that they are always diagonalisable in an orthonormal basis. A similar result can be obtained in our case, but with an important difference. The mere fact that $D\Phi$ is symmetric is not enough to Φ being diagonalisable. An example of this situation is given by the function $\Phi(x,y) = (x+y, y^2+x)$. But assuming also that Φ is diagonalisable gives the desired result.

Proposition 4. *Let $\Phi\colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^1$ function. If Φ is diagonalizable and $D\Phi$ is symmetric, then there exists an orthogonal matrix Q such that $\Psi := Q^t \circ \Phi \circ Q$ is diagonal.*

The converse result also holds.

Proposition 5. *Let $\Phi\colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^1$ function. Suppose there exists an orthogonal matrix Q that diagonalizes Φ. Then the Jacobian matrix $D\Phi$ is symmetric.*

Applications

It is known that the diagonal representation of linear maps allows the study of several problems, such as the computation of higher order compositions or the inversion of the map, among others in the field of differential equations. But this problems can also be present in nonlinear contexts. For instance, in the following autonomous system of Ordinary Differential Equations (ODE) for $x(t), y(t)$:

$$\begin{aligned} \frac{dx}{dt} &= 2e^{-x+y} - x + 2y \\ \frac{dy}{dt} &= e^{-x+y} - x + 2y. \end{aligned}$$

We can write it as $\frac{dX}{dt} = \Phi(X)$, where $X(t) := (x(t), y(t))$ and

$$\Phi(X) = \Phi(x,y) = (2e^{-x+y} - x + 2y, e^{-x+y} - x + 2y).$$

Applying Theorem 2 we get $\Delta = e^{-2x}(e^x + e^y)^2 \geq 0$ and $A = 7/10, B = 9/100$. So, this function is diagonalisable, and the matrices of the linear isomorphism that diagonalises Φ and its inverse can be

$$P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

Then the diagonal expression is $\Psi(p,q) := (e^{-p}, q)$, and the relation between variables are $p = x - y$ and $q = -x + 2y$. Thus, the original systems is transformed to

$$\begin{cases} \frac{dp}{dt} = e^{-p} \\ \frac{dq}{dt} = q, \end{cases}$$

than can be easily solve as two ODEs. For the second one we obtain $q(t) = K_2 e^t$. The first one is also a separable equation that can be solved as $p(t) = \log(t + K_1)$. So, the solutions for x and y are $x(t) = 2\log(t + K_1) + K_2 e^t$ and $y(t) = \log(t + K_1) + K_2 e^t$. On the other hand, in some contexts is useful to invert the function Φ, see [3, Section 6.2.1]. In our example the diagonal function Ψ can be easily inverted as $\Psi^{-1}(p,q) = (-\log(p), q)$, and so

$$\begin{aligned} \Phi^{-1}(x,y) &= (P \circ \Psi^{-1} \circ P^{-1})(x,y) \\ &= (P \circ \Psi^{-1})(x - y, -x + 2y) \\ &= P \begin{pmatrix} -\log(x-y) \\ -x + 2y \end{pmatrix} = \begin{pmatrix} -2\log(x-y) - x + 2y \\ -\log(x-y) - x + 2y \end{pmatrix} \end{aligned}$$

Conclusions and further work

Although the conditions for a non-linear function to be diagonalised are restrictive, when it is possible we obtain a powerful technique that allows us to tackle complex problems. For example, there are problems of current interest in Random Differential Equations which until now could only be tackled by numerical approximations, and which can now be solved exactly.
As future work it remains to obtain characterisations like the one in Theorem 2 for the case of $\mathbb{R}^3$ or even to generalise it to $\mathbb{R}^n$. Moreover, it can be further explored the concepts of eigenvector and eigenfunctions. For instance, the determinant of a nonlinear map can be defined by the product of its eigenfunctions and its trace by the sum. It can be interesting finding the relation of this concepts with the original map.

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